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LETTER TO THE EDITOR

A note on graded Yang–Baxter solutions as braid-monoid invariants

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Abstract. We construct two $Osp(n|2m)$ solutions of the graded Yang–Baxter equation by using the algebraic braid-monoid approach. The factorizable S-matrix interpretation of these solutions is also discussed.

It is known that the Yang–Baxter equations play a central role in the study of two-dimension exactly solvable models [1–3]. One possible generalization of the Yang–Baxter relation is to consider integrable systems containing both bosonic and fermionic degrees of freedom. In this case the elementary generator A_i acts on a Z_2 graded vector space and its bosonic and fermionic components are distinguished by the parity $p(A_i) \equiv p(i) = 0, 1$, respectively [2]. Considering a graded space $V^{(n|m)}$ consisted of n bosons and m fermions, the graded Yang–Baxter equation for the R-matrix amplitude is written as [2]

$$R_{a_1,a_2}^{\alpha,\gamma}(u)R_{\alpha,a_3}^{b_1,\delta}(v+u)R_{\gamma,\delta}^{b_2b_3}(v)(-1)^{p(\gamma)[p(a_3)+p(\delta)]}$$
$$= R_{a_2,a_3}^{\gamma,\delta}(v)R_{a_1,\delta}^{\alpha,b_3}(v+u)R_{\alpha,\gamma}^{b_1,b_2}(u)(-1)^{p(\gamma)[p(b_3)+p(\delta)]} \tag{1}$$

where $p(i) = 0$ for $i = 1, 2, \ldots, n$; $p(i) = 1$ for $i = n+1, n+2, \ldots, n+m$. It has also been assumed that the non-null elements $R_{a,b}^{c,d}$ are commuting variables, namely $p(R_{a,b}^{c,d}) = 0$ [2]. The simplest solution of the graded Yang–Baxter equation (1) was exhibited by Kulish and Sklyanin [2] as the generalization of Yang's S-matrix [4], and is given by

$$R(u,\eta) = \frac{u}{u+\eta}I + \frac{\eta}{u+\eta}P^{g} \tag{2}$$

where I is the identity and P^{g} is the graded permutation operator on the tensor vector space $V^{(n|m)} \otimes V^{(n|m)}$ with elements $(P^{g})_{a,b}^{c,d} = (-1)^{p(a)p(b)}\delta_{a,d}\delta_{b,c}$. Variable u is the spectral parameter while η is a constant connected to the graded classical solution [2].

In general, solutions of the *graded* Yang–Baxter relation have been investigated as invariants under the superalgebras $Sl(n|m)$ and $Osp(n|2m)$ [5, 6, 7]. For instance, trigonometric solutions have been constructed by Bazhanov and Shadrikov [5] by investigating the classical analogue of the graded Yang–Baxter equations. The special case of the universal $Osp(2|1)$ R-matrix was discussed in [6, 7] in the context of the quantum super-group. Nowadays, however, it has been recognized that the Yang–Baxter solutions are deeply connected to a number of other algebraic structures, e.g., the braid-monoid [8], the Temperley–Lieb (TL) [9] algebras and more recently the multi-colour versions of these

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structures [10]. In this sense one would expect that similar relations would also appear for the *graded* case. In fact, Deguchi and Akutsu [11] have shown that the fundamental $Sl(n|m)$ graded solution can be obtained through the generators of the Hecke algebra. Motivated by this fact, the purpose of this note is to discuss two $Osp(n|2m)$ solutions generated by the braid-monoid invariants. We also comment on the crossing symmetry property which is fundamental in the context of factorizable S-matrices interpretation of our solutions.

We start our discussion by constructing an $Osp(n|2m)$ TL invariant operator. In order to build-up such an operator we recall that an $Osp(n|2m)$ invariant A is a $(n+2m)X(n+2m)$ matrix satisfying the property (see, e.g., [12])

$$A+\alpha A^{\mathrm{st}}\alpha^{-1}=0 \tag{3}$$

where the symbol A^{st} denotes the supertranspose operation in the matrix A and the matrix α is given by

$$\alpha=\begin{pmatrix} I_{nXn} & O_{nX2m} \\ O_{2mXn} & \begin{pmatrix} O_{mXm} & I_{mXm} \\ -I_{mXm} & O_{mXm} \end{pmatrix} \end{pmatrix} \tag{4}$$

where $I_{aXa}(O_{aXa})$ is the aXa identity (null) matrix. Remarkably enough, we notice that the matrix α present in $Osp(n|2m)$ invariance plays a fundamental role on the construction of our TL invariant. Indeed, if we define the following generator E_i as

$$E_i=\sum_{abcd}\alpha_{ab}\alpha^{\mathrm{st}}_{cd}e^{i}_{ac}\otimes e^{i+1}_{bd} \tag{5}$$

one can check that the Temperley–Lieb relations are satisfied, namely

$$E_iE_{i\pm1}E_i=E_i \qquad E_i^2=(n-2m)E_i \qquad [E_i,E_j]=0 \qquad \text{for } |i-j|\geqslant 2 \tag{6}$$

where the matrix elements of e^i_{ab} acting on the ith 'site' are $(e^i_{ab})_{cd}=\delta_{a,c}\delta_{b,d}$.

The next step is to show how one can *graded* 'Baxterize' the explicit representation (5) for the monoid E_i. However, from the discussions of [2, 11] we notice that a null-parity graded R-matrix satisfying (1) can be obtained by the relation

$$R_i(u)=P_i^gX_i(u) \tag{7}$$

where $X_i(u)$ is a *null-parity* usual Yang–Baxter operator satisfying the relation

$$X_i(u)X_{i+1}(u+v)X_i(v)=X_{i+1}(v)X_i(u+v)X_{i+1}(u). \tag{8}$$

Finally, taking into account the previous experience [13, 14] in the Baxterization of a TL generator we find the following solution

$$R_i(u,\eta)=P_i^g+f(u,\eta)E_i \tag{9}$$

where we have used the fact that $p(E_i)=0$ and the important identity

$$P_i^gE_i=E_iP_i^g=E_i \tag{10}$$

and function $f(u,\eta)$ is given by

$$f(u,\eta)=\begin{cases}\pm\dfrac{\sinh(u/\eta)}{\sinh(\gamma-u/\eta)} & \text{if } 2\cosh(\gamma)=(n-2m)\lesseqgtr\pm 2\\ \pm\dfrac{u}{\eta-u} & \text{if } n-2m=\pm 2\\ \dfrac{\sin(u/\eta)}{\sin(\gamma-u/\eta)} & \text{if } 2\cos(\gamma)=|n-2m|<2.\end{cases}\tag{11}$$

We would like to stress that one advantage of this approach is that we are able to generate a new trigonometric/rational $Osp(n|2m)$ solution which has no *graded* classical analogue, and apparently for this reason has been missed in the literature [5]. From the point of view of quantum spin chains the operator E_i generalizes previous efforts in finding isotropic high-spins [15, 16] TL invariants. The fact that the TL parameter $(n-2m)$ may assume negative values means that an appropriate deformation of isotropic high-spin chains [17, 18] shall indeed possess a hidden $Osp(n|2m)$ symmetry. For instance, we have checked that the simplest case of $Osp(1|2)$ model corresponds to the deformed point $q=i$ (in the notation of [17]) of the spin-1 TL chain.

A second important feature of this approach is as follows. First of all, identity (10) strongly suggests that the operators P_i^g and E_i may be generators of a more general algebraic structure, namely the braid-monoid algebra. Moreover, taking into account the remarks of [8], one can verify that a crossing symmetric S-matrix interpretation of (9) will lead us in the high-energy limit to the braid operator P_i^g and at the crossing point $u=\eta\gamma$ to the monoid operator E_i. More precisely, one can show that besides equation (10) we have the following extra relations

$$P_{i\pm1}^g P_i^g E_{i\pm1}=E_i P_{i\pm1}^g P_i^g=E_i E_{i\pm1}\tag{12}$$

$$E_i P_{i\pm1} E_i=E_i\tag{13}$$

and the braid-inverse properties

$$P_{i+1}^g P_i^g P_{i+1}^g=P_i^g P_{i+}^g P_i^g\tag{14}$$

$$P_i^g P_i^g=I_i.\tag{15}$$

In fact we can show that these sets of relations between the operators P_i^g and E_i† form a degenerated representation of a reduced‡ Birman–Wenzel algebra (see e.g. [8, 20]). It is also possible to show that the other relations between the operators P_i^g and E_i closing the reduced Birman–Wenzel algebra are just a consequence of the identities (12, 13, 15). Hence, this observation suggests that another *graded* Baxterization can be implemented in the sense of that found by Jones [19]. Therefore, proceeding as in the TL case and taking as a guess the Jones [19] parametrization of the degenerated point of the Birman–Wenzel algebra [20] we find that

$$R_i(u,\eta)=\frac{u}{\eta}I_i+P_i^g\frac{u}{u+\eta(n-2m-2)/2}E_i\tag{16}$$

† At this point we remind the reader that more general forms of such monoid can be chosen. For instance, we mention the monoid $E_i=\sum_{abcd}\alpha_{ab}\alpha_{cd}^{-1}e_{ac}^i\otimes e_{cd}^{i+1}$ where $\alpha=\mathrm{diag}[A_{nXn}, \mathrm{antdiag}(B_{mXm}, -B_{mXm})]$ if A and B are symmetric and invertible matrices.

‡ This occurs at the singular point of the parameters entering the Birman–Wenzel algebra such that the eigenvalues become degenerated.

satisfies the *graded* Yang–Baxter equation (1). A simple way to verify this last result is by checking that $P_i^g R_i(u,\eta)$ satisfies the usual Yang–Baxter equation (8) if one uses the relations (6, 10, 12–15). We recall, however, that this solution corresponds to the rational limit of a trigonometric $Osp(n|2m)$ solution already found by Bazhanov and Shadrikov [5]. This is due to the fact that (16) admits its *graded* classical analogue around the point $1/\eta \simeq 0$.

To conclude we would like to make some remarks concerning the interpretation of the graded solutions (9, 16) as factorizable S-matrices. In order to interpret $R_i(u,\eta)$ as a S-matrix one has to impose crossing and unitarity conditions. Although the unitarity condition remains as usual, the crossing property in the graded case now has to take into account the signs coming from the interchange of two fermions. This is accomplished by taking the supertranspose instead of the traditional transpose operation, and the crossing symmetry property becomes

$$S_i(\theta) = C \otimes I S_i^{\mathrm{st}_i}(\mathrm{i}\pi - \theta)(C \otimes I)^{\mathrm{st}_i} \tag{17}$$

where θ is the relativistic rapidity and the supertranspose is taken only on the first space of $S_i(\theta)$. C is the charge matrix, which for the theories (9, 16) is $C = \alpha$. After some calculations, the corresponding S-matrix associated to the solution (9) is given by

$$S_i(\theta) = f(\theta) \sin\left(\frac{\pi - \mathrm{i}\theta}{\eta}\right) R(\mathrm{i}\theta, \eta) \qquad |n - 2m| = 2\cos(\pi/\eta) \tag{18}$$

$$f(\theta) = f(\mathrm{i}\pi - \theta) \qquad f(\theta)f(-\theta) = \left[\sin\left(\frac{\pi - \mathrm{i}\theta}{\eta}\right)\sin\left(\frac{\pi + \mathrm{i}\theta}{\eta}\right)\right]^{-1} \tag{19}$$

and for solution (16) we have

$$S_i(\theta) = f(\theta) R_i\left(\mathrm{i}\theta, \eta = \frac{2\pi}{n - 2m - 2}\right) \qquad f(\theta)f(-\theta) = \frac{\theta^2}{\theta^2 + \eta^2}. \tag{20}$$

The main feature of these S-matrices is that they have a formally remarkable resemblance to those describing the physics of $O(N)$ invariant systems [21, 22]. Indeed, at $m = 0$ the $O(n)$ symmetry is automatically restored in the solutions (18–20). The physical interpretation of these solutions is as follows. The first solution can be considered as a regularized version of that proposed by Zamolodchikov [21] to describe the physics of a self-avoiding polymer. In our case, however, we can choose $n, m \neq 0$ such that the self-avoiding limit $\eta = 0$ is taken in an unambiguous way. The second solution (20) generalizes the S-matrices corresponding to the $O(N)$ nonlinear sigma model [22]. An important feature is that now the simplest case of $Osp(1|2)$ has its pole on the physical strip at $\theta = \mathrm{i}2\pi/3$ (in function $f(\theta)$) which is not present in its equivalent bosonic version, namely the $O(3)$ nonlinear sigma case. We believe that this is a very interesting solution and we hope to discuss its other features, e.g., the associated quantum spin chain and the quantum field theory, in a further publication.

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